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ORIGINAL PAPER

Dynamical analysis of a stochastic model for cascaded continuous flow bioreactors

Tonghua Zhang · Zhenzhen Chen · Maoan Han

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Abstract In this paper, the long time behavior of a stochastic model is studied when the Contois growth rate is employed in reactor cascades. We first investigate the existence and uniqueness of the positive solution of the model. Then it is followed by the stochastic stability analysis of the equilibria, which is based on the so-called Lyapunov function. Our study shows that under certain condition, both the washout and non-washout equilibria are stochastically stable. At the end of this paper, numerical simulations are carried out to illustrate our theoretical results.

1 Introduction

Waste water is a complex mixture of biodegradable organic materials such as substrates and microorganisms. The biological treatment of waste water is a method of using microorganisms to remove substrates or pollutants which can harm the environment, and has been long used. In order to deeply understand such processes, mathematical models have been developed, see [1–6] for example, where the deterministic models have been employed to describe the processes. Denote the concentrations of the substrate and microorganism in bio-reactor by $S(t)$ and $X(t)$. Then we have a model in a form of the following

$$\begin{cases} \dot{S} = \frac{1}{\tau}(S_0 - S) - \frac{1}{\alpha} X g(S, X), \\ \dot{X} = \frac{1}{\tau}(X_0 - X) + X g(S, X) - k_d X. \end{cases}$$

T. Zhang · Z. Chen
Department of Mathematics, Swinburne University of Technology, Hawthorn, VIC 3122, Australia
e-mail: tonghuazhang@swin.edu.au

M. Han (✉)
Department of Mathematics, Shanghai Normal University, Shanghai, China
e-mail: mahan@shnu.edu.cn

In order to improve the efficiency of the waste water treatment, some researcher proposed to connect few reactors in series, which results in reactor cascades [7]. This process can be governed by the following mathematical model.

$$\begin{cases} \dot{S}_i = \frac{1}{\tau}(S_{i-1} - S_i) - \frac{1}{\alpha} X_i g(S_i, X_i), \\ \dot{X}_i = \frac{1}{\tau}(X_{i-1} - X_i) + X_i g(S_i, X_i) - k_d X_i, \end{cases} \tag{1.1}$$

where S_i and X_i are concentrations of the substrate and microorganism in reactor n; functions $g(S_i, X_i), i = 1, 2, \ldots, n$ are the rates of growth of microorganisms, also known as response functions in biological models. For example Monod's growth rate

$$g(S, X) = \frac{\mu_m S}{K_s + S},$$

was used by Sonmezisik et al. [3] and Zhang [5]; Contois' growth rate

$$g(S, X) = \mu_{\max} \left(\frac{S/X}{K_x + S/X} \right)$$

was used in [3,8] and Tessier's growth rate

$$g(S, X) = \mu_m \left(1 - \exp\left(-\frac{S}{K_s} \right) \right)$$

was used in [1–4]. Which form should be used in a model depends on the reaction kinetics. It has been suggested that the Contois specific growth rate accurately describes experimental data when mass-transfer limitations ensure that the underlying kinetic process is restricted by the available surface area. If the Contois specific growth rate is selected, after dimensionlization, the function takes the form of $g(S_i, X_i) = \frac{S_i}{S_i + X_i}$.

In this paper, assume due to a noise from the environment, parameter k_d is perturbed. Furthermore, we assume that the growth medium fed into the bioreactor is sterile, i.e. there are no microorganism in the first reactor at time $t = 0$ and the dimensionless substrate concentration in the feed is normalized. In other words, in the rest of this paper, we make the following assumption

$$X_0 = 0, S_0 = 1, \quad k_d \to k_d + \sigma dB(t),$$

where σ is the noise intensity. Then we reach a mathematical model

$$\begin{cases} \dot{S}_i = \frac{1}{\tau}(S_{i-1} - S_i) - \frac{1}{\alpha} X_i g(S_i, X_i), \\ \dot{X}_i = \frac{1}{\tau}(X_{i-1} - X_i) + X_i g(S_i, X_i) - k_d X_i - \sigma X_i \frac{dB(t)}{dt}, \quad i = 1, 2, \ldots, n. \end{cases} \tag{1.2}$$

Please notice this is a stochastic model in the Itô form and when $\sigma = 0$ (1.2) is the deterministic model investigated in [7]. Some routine analysis suggest that the ith

reactor of the model has at most two equilibria: a washout equilibrium $E_i(1, 0)$ and a non-washout equilibrium

$$E_i^* = \left(S_i^*, X_i^*\right),$$

where

$$\begin{aligned}
X_i^* &= \frac{-b_i - \sqrt{b_i^2 - 4a_i c_i}}{2a_i},\\
A_i &= S_{i-1}^* + \frac{X_{i-1}^*}{\alpha},\\
B_i &= \frac{1 + k_d \tau}{\alpha},\\
a_i &= (B_i - 1)k_d \tau + B_i(1 - \tau) - 1,\\
b_i &= (1 - B_i)X_i^* + A_i[(1 - k_d)\tau - 1],\\
c_i &= A_i X_{i-1}^*,\\
S_i^* &= A_i - B_i X_i^*.
\end{aligned}$$

Notice that $0 < k_d < 1$, from [7] we know that the non-washout equilibrium E_i^* is physically meaningful if and only if $\tau > \frac{1}{1-k_d}$. The washout branch is stable when the non-washout branch is not physically meaningful ($\tau < \frac{1}{1-k_d}$). The non-washout branch is stable when it is physically meaningful. In this paper, we investigate the effect of the random perturbation on the stability of the equilibria and also the existence of positive solution.

The rest of this paper is organized as follows. In Sect. 2, we show that a unique positive solution exists no matter how large the intensities of noises are in the stochastic model. In Sect. 3, we prove that the washout equilibrium E_i in each reactor is stochastically asymptotically stable under certain condition. In Sect. 4, we shall show the stochastic stability of the non-washout equilibrium of the first reactor. Also we show that if the noise is large enough, it can lead to washout in the cascade. In Sect. 5 we conclude the paper with numerical simulations.

2 Existence and uniqueness of the positive solution of model (1.2)

In this section we shall prove that model (1.2) has a unique positive solution for given positive initial values. It is summarised in the following theorem.

Theorem 2.1 *Given initial value* $(S(0), X(0)) \in R_+^2 = \{(x_1, x_2) \in R^2 : x_i > 0, i = 1, 2\}$, *the model* (1.2) *has a unique solution*

$$(S_1(t), X_1(t), S_2(t), X_2(t), \ldots, S_n(t), X_n(t))$$

for $t \in [0, \infty)$. *Furthermore, the solution will remain in* R_+^{2n} *with probability 1, namely* $(S_1(t), X_1(t), S_2(t), X_2(t), \ldots, S_n(t), X_n(t)) \in R_+^{2n}$ *for all* $t \geq 0$ *almost surely (a.s.).*

Proof It is easy to verify that the coefficients of model (1.2) are locally Lipschitz continuous. Hence, for any given value $(S(0), X(0)) \in R_+^2$, there is a unique local solution $(S_1(t), X_1(t), S_2(t), X_2(t), \ldots, S_n(t), X_n(t))$ for t on $[0, \tau_e)$, where τ_e is the explosion time.

Next, we can actually prove that $\tau_e = \infty$, which implies the solution is global *a.s.*. Assume $m_0 > 0$ is large enough so that

$$S(0) \in \left[\frac{1}{m_0}, m_0\right], X(0) \in \left[\frac{1}{m_0}, m_0\right]$$

and for each integer $m \geq m_0$, define the stopping time as follows

$$\tau_m = \inf\left\{t \in [0, \tau_e) : S_i(t) \notin \left(\frac{1}{m}, m\right) \text{ or } X_i(t) \notin \left(\frac{1}{m}, m\right)\right\},$$

where $i = 1, 2, 3, \ldots, n$. For the sake of completeness, for empty set, $\emptyset$, we define the stopping time as $\inf \emptyset = \infty$. Note that τ_m is increasing as $m \to \infty$ and $\tau_\infty = \lim_{m\to\infty} \tau_m \leq \tau_e$ *a.s.*. Then in the rest of this section, we only need to demonstrate that $\tau_\infty = \infty$ *a.s.*.

By the proof of contradiction, if this statement is not true, then for any given $T > 0$ there is a $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon \text{ for all } m \geq m_1. \tag{2.1}$$

Let $f(u) = u - 1 - \ln u$ for $u \geq 0$. It is easy to verify that $f(0) = \lim_{u\to 0} f(u) > 0$, $f(\infty) = \lim_{u\to\infty} f(u) > 0$ and

$$f'(u) = 1 - \frac{1}{u} = \begin{cases} < 0, & \text{for } 0 \leq u < 1, \\ = 0, & \text{for } u = 1, \\ > 0, & \text{for } u > 1, \end{cases}$$

which implies $f(u) = 0$ if and only if $u = 1$. If on R_+^2, define the Lyapunov function

$$V(S_1, X_1, S_2, X_2, \ldots, S_n, X_n) = \sum_{i=1}^{n} \alpha(S_i - 1 - \ln S_i) + \sum_{i=1}^{n}(X_i - 1 - \ln X_i).$$

Then $V(S_1, X_1, S_2, X_2, \ldots, S_n, X_n) \geq 0$, since $\alpha > 0$. Using Itô's formula yields

$$\begin{aligned} dV &= \sum_{i=1}^{n} \alpha \left[dS_i - \frac{1}{S_i} dS_i + \frac{1}{2S_i^2}(dS_i)^2\right] + \sum_{i=1}^{n}\left[dX_i - \frac{1}{X_i} dX_i + \frac{1}{2X_i^2}(dX_i)^2\right] \\ &= LV dt + \sum_{i=1}^{n}(1 - X_i)\sigma dB(t), \end{aligned}$$

where

$$
\begin{aligned}
LV &= \sum_{i=1}^{n}\Bigg(\frac{\alpha}{\tau}(S_{i-1}-S_i)-\frac{X_iS_i}{X_i+S_i}-\frac{\alpha}{\tau S_i}(S_{i-1}-S_i)+\frac{X_i}{X_i+S_i}+\frac{1}{\tau}(X_{i-1}-X_i)\\
&\quad+\frac{X_iS_i}{X_i+S_i}-k_dX_i-\frac{1}{\tau X_i}(X_{i-1}-X_i)-\frac{S_i}{X_i+S_i}+k_d+\frac{1}{2}\alpha^2\Bigg)\\
&\le \sum_{i=1}^{n}\left(\frac{\alpha}{\tau}(S_{i-1}-S_i)-\frac{\alpha}{\tau S_i}S_{i-1}+\frac{\alpha}{\tau}+\frac{X_i}{X_i+S_i}+\frac{1}{\tau}(X_{i-1}-X_i)+\frac{1}{\tau}+k_d+\frac{1}{2}\alpha^2\right)\\
&\le \sum_{i=1}^{n}\left(\frac{\alpha}{\tau}(S_{i-1}-S_i)+\frac{1}{\tau}(X_{i-1}-X_i)\right)+n\left(\frac{1}{\tau}+\frac{\alpha}{\tau}+1+k_d+\frac{1}{2}\alpha^2\right)\\
&\le \left(\frac{\alpha}{\tau}(S_0-S_n)+\frac{1}{\tau}(X_0-X_n)\right)+n\left(\frac{1}{\tau}+\frac{\alpha}{\tau}+1+k_d+\frac{1}{2}\alpha^2\right)\\
&\le \left(\frac{\alpha}{\tau}S_0+\frac{1}{\tau}X_0\right)+n\left(\frac{1}{\tau}+\frac{\alpha}{\tau}+1+k_d+\frac{1}{2}\alpha^2\right)\triangleq K.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^{\tau_m\wedge T} dV(S_1(r),X_1(r),S_2(r),X_2(r),\ldots,S_n(r),X_n(r))\\
&\quad\le \int_0^{\tau_m\wedge T} K\,dr+\int_0^{\tau_m\wedge T}\sum_{i=1}^{n}(1-X_i)\sigma\,dB(t).
\end{aligned}
$$

Taking expectation on both sides of the above inequality yields

$$
\begin{aligned}
&E[V(S_1(\tau_m\wedge T),X_1(\tau_m\wedge T),\ldots,S_n(\tau_m\wedge T),X_n(\tau_m\wedge T))]\\
&\quad\le V(S(0),X(0))+E\int_0^{\tau_m\wedge T}K\,dr\le V(S(0),X(0))+KT.
\end{aligned}
\tag{2.2}
$$

For integer $m\ge m_1$, denote the set of $\{\tau_m\le T\}$ by Ω_m. It then follows from (2.1) that $P(\Omega_m)\ge\varepsilon$. It implies that for each $\omega\in\Omega_m$, we have at least one of $S_i(\tau_m,\omega)$, $X_i(\tau_m,\omega)$ equal to either m or $\frac{1}{m}$. Either case will give the following relation:

$$
\begin{aligned}
&V(S_1(\tau_m\wedge T),X_1(\tau_m\wedge T),\ldots,S_n(\tau_m\wedge T),X_n(\tau_m\wedge T))\\
&\quad\ge \alpha(m-1-\ln m)\wedge(m-1-\ln m)\wedge\alpha\left(\frac{1}{m}-1-\ln\frac{1}{m}\right)\wedge\left(\frac{1}{m}-1-\ln\frac{1}{m}\right),
\end{aligned}
$$

which together with equations (2.1) and (2.2) yields

$$\begin{aligned} &V(S(0), X(0)) + KT \\ &\geq E[\mathbf{1}_{\Omega_m(\omega)} V(S_1(\tau_m \wedge T), X_1(\tau_m \wedge T), \ldots, S_n(\tau_m \wedge T), X_n(\tau_m \wedge T))] \\ &\geq \alpha(m - 1 - \ln m) \wedge (m - 1 - \ln m) \wedge \alpha \left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right) \wedge \left(\frac{1}{m} - 1 - \ln \frac{1}{m}\right). \end{aligned} \tag{2.3}$$

Here $\mathbf{1}_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Please note on one hand the left hand side of (2.3) is independent of m and finite, on the other hand letting $m \to \infty$ leads to the right hand side infinity. This is a contradiction. Thus $\tau_\infty = \infty$ $a.s.$. □

3 Asymptotical stability of the washout equilibrium

From Sect. 1 we know that for the deterministic model (1.1), there always exists one stable washout equilibrium $E_1 = (S_1, X_1, S_2, X_2, \ldots, S_n, X_n) = (1, 0, 1, 0, \ldots, 1, 0)$ when $\tau < \frac{1}{1-k_d}$. And obviously it is still a equilibrium of the stochastic model (1.2). In this section, our intention is to investigate the stochastic effects on the stability of E_1. Our main result will be stated in Theorem 3.3. In order to prove our main result here, we need some preliminaries from previous work, for example [9], based on which we list these preliminaries as two lemmas below for self-contained.

Assume an n-dimensional differential equation has the form

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \tag{3.1}$$

where $B(t)$ is an m-dimensional Brownian motion and $f(t, X(t)) : [t_0, +\infty) \times R^n \to R^n$, $g(t, X(t)) : [t_0, +\infty) \times R^n \to R^{n\times m}$. And we assume that $f(t, X(t))$ and $g(t, X(t))$ satisfies the local Lipschitz condition. The linearization of (3.1) is

$$dX(t) = F(t, X(t))dt + G(t, X(t))dB(t). \tag{3.2}$$

By establishing Lyapunov functions, we can obtain the following lemma, which provides conditions for various types of stability for linear stochastic model (3.2).

Lemma 3.1 *Suppose that there exists a non-negative function $V(t, x) \in C^2(R \times R)$, two continuous functions $a(\cdot)$ and $b(\cdot)$ and a positive constant K such that, for $|x| < K$,*

$$a(|x|) \leq V(t, x) \leq b(|x|).$$

We have

- (i) *if $LV \leq 0$, $x \in [0, K]$, then the solution of (3.2) is stable in probability 1.*
- (ii) *if there exists a continuous function $C : R_+ \to R_+$, such that $LV \leq -C(|x|)$, then the trivial solution of (3.2) is stochastically asymptotically stable.*

Many problems concerning the stability of the equilibrium in a nonlinear stochastic model can be reduced to problems concerning the stability of solutions in the associated linear model.

Lemma 3.2 ([9] Theorem 2.2) *If linear model* (3.2) *with constant coefficients* ($F(t) = F, G(t) = G$) *is stochastically asymptotically stable, and the coefficients of* (3.1) *and* (3.2) *satisfy the inequality*

$$|f(t, X) - F \cdot X| + |g(t, X) - G \cdot X| < \delta |X|$$

in a sufficiently small neighborhood of the point $X = 0$ *and with a sufficiently small constant* δ*, then the trivial solution* $X(t) = 0$ *of* (3.1) *is stochastically asymptotically stable.*

We now give sufficient conditions for the stability of the stochastic model (1.2)

Theorem 3.3 *When* $\tau < \frac{1}{1-k_d}$*, the washout equilibrium* $E = (S_1, X_1, S_2, X_2, \ldots, S_n, X_n) = (1, 0, 1, 0, \ldots, 1, 0)$ *of model* (1.2) *is stochastically asymptotically stable if the intensity of the noise,* σ *satisfies*

$$\sigma^2 \le \frac{2}{\tau} - 2 + 2k_d.$$

Proof Consider the model equations for the first reactor of the cascade system (1.2)

$$\begin{cases} \dot{S}_1 = \dfrac{1}{\tau}(S_0 - S_1) - \dfrac{1}{\alpha}\dfrac{X_1 S_1}{X_1 + S_1}, \\ \dot{X}_1 = \dfrac{1}{\tau}(X_0 - X_1) + \dfrac{X_1 S_1}{X_1 + S_1} - k_d X_1 - \sigma X_1 \dfrac{dB(t)}{dt}. \end{cases} \tag{3.3}$$

We next prove that the washout equilibrium $E_1 = (S_1, X_1) = (1, 0)$ of the first reactor is stochastically asymptotically stable when $\sigma^2 \le \frac{2}{\tau} - 2 + 2k_d$. It will imply the washout equilibrium $E = (S_1, X_1, S_2, X_2, \ldots, S_n, X_n) = (1, 0, 1, 0 \ldots, 1, 0)$ of the model (1.2) is stochastically asymptotically stable.

First, performing a transformation $Z_1(t) = S_1(t) - S_0$, $Z_2(t) = X_1(t)$ changes model (3.3) into

$$\begin{cases} \dot{Z}_1 = -\dfrac{1}{\tau} Z_1 - \dfrac{1}{\alpha}\dfrac{Z_2(Z_2 + S_0)}{Z_2 + Z_1 + S_0}, \\ \dot{Z}_2 = -\dfrac{1}{\tau} Z_2 + \dfrac{Z_2(Z_2 + S_0)}{Z_2 + Z_1 + S_0} - k_d Z_2 - \sigma Z_2 \dfrac{dB(t)}{dt}. \end{cases} \tag{3.4}$$

Linearizing it at the origin (0, 0), then we get

$$\begin{cases} \dot{Z}_1 = -\frac{1}{\tau} Z_1 - \frac{1}{\alpha} Z_2, \\ \dot{Z}_2 = -\frac{1}{\tau} Z_2 + Z_2 - k_d Z_2 - \sigma Z_2 \frac{dB(t)}{dt}. \end{cases} \tag{3.5}$$

Our proof will then be split into 2 steps: firstly, to prove the stability of the equilibrium of (3.5), and secondly, to prove the stability of the equilibrium of (3.4). Following this idea, we next show the equilibrium $(Z_1, Z_2) = (0, 0)$ of model (3.5) is stochastically asymptotically stable. To this end we define a Lyapunov function, V, by

$$V = Z_1^2 + Z_2^2 + AZ_2$$

with A is a positive constant to be selected according to our needs. Along the trajectories of model (3.5) we have

$$\begin{aligned} dV &= 2Z_1 dZ_1 + (dZ_1)^2 + 2Z_2 dZ_2 + (dZ_2)^2 + AdZ_2 \\ &= LVdt + \left(-2\sigma Z_2^2 - A\sigma Z_2\right) dB(t), \end{aligned}$$

where

$$\begin{aligned} LV &= -\frac{2}{\tau}Z_1^2 - \frac{2}{\alpha}Z_1 Z_2 - \frac{2}{\tau}Z_2^2 + 2Z_2^2 - 2k_d Z_2^2 + \sigma^2 Z_2^2 + \left(-\frac{A}{\tau} + A - Ak_d\right) Z_2 \\ &= -\frac{2}{\tau}Z_1^2 - \frac{2}{\alpha}Z_1 Z_2 + \left(-\frac{2}{\tau} + 2 - 2k_d + \sigma^2\right) Z_2^2 + A\left(-\frac{1}{\tau} + 1 - k_d\right) Z_2. \end{aligned}$$

We claim that LV is a negative definite, which implies the $(0, 0)$ origin of (3.5) is globally stochastic asymptotically stable. In fact,

(i) If $Z_1 \geq 0$, then

$$LV \leq -\frac{2}{\tau}Z_1^2 + \left(-\frac{2}{\tau} + 2 - 2k_d + \sigma^2\right) Z_2^2.$$

Obviously we can get $LV \leq 0$ when $\sigma^2 \leq \frac{2}{\tau} - 2 + 2k_d$. And $LV = 0$ if and only if $Z_1 = Z_2 = 0$.

(ii) If $Z_1 < 0$, we can get $-Z_1 < S_0$ from $S_1 = Z_1 + S_0 > 0$, then

$$LV \leq -\frac{2}{\tau}Z_1^2 + \frac{2}{\alpha}S_0 Z_2 + \left(-\frac{2}{\tau} + 2 - 2k_d + \sigma^2\right) Z_2^2 + A\left(-\frac{1}{\tau} + 1 - k_d\right) Z_2.$$

In this case, choosing

$$A = \frac{\frac{2}{\alpha}S_0}{-\frac{2}{\tau} + 2 - 2k_d}$$

yields

$$LV \leq -\frac{2}{\tau}Z_1^2 + \left(-\frac{2}{\tau} + 2 - 2k_d + \sigma^2\right) Z_2^2.$$

Again, when $\sigma^2 \le \frac{2}{\tau} - 2 + 2k_d$, $LV \le 0$, and $LV = 0$ if and only if $Z_1 = Z_2 = 0$. So far we have proved that LV is negative definite. Next, we show that the origin (0,0) of model (3.4) is also stochastically asymptotically stable, which implies the washout equilibrium $E_1(1, 0)$ of model (3.3) is stochastically asymptotically stable. Notice

$$\begin{aligned} &|f(t, X) - F \cdot X| + |g(t, X) - G \cdot X| \\ &= \sqrt{\left(\frac{1}{\alpha}Z_2 - \frac{1}{\alpha}\frac{Z_2(Z_1 + S_0)}{Z_2 + Z_1 + S_0}\right)^2 + \left(\frac{Z_2(Z_1 + S_0)}{Z_2 + Z_1 + S_0} - Z_2\right)^2} \\ &= \sqrt{1 + \frac{1}{\alpha^2}}\left|Z_2 - \frac{Z_2(Z_1 + S_0)}{Z_2 + Z_1 + S_0}\right| \\ &= \sqrt{1 + \frac{1}{\alpha^2}}\left|\frac{Z_2^2}{Z_2 + Z_1 + S_0}\right|. \end{aligned}$$

For small $\varepsilon > 0$, when $Z_1 < \varepsilon, Z_2 < \varepsilon$, choose $l = \sqrt{1 + \frac{1}{\alpha^2}}$, it follows that

$$|f(t, X) - F \cdot X| + |g(t, X) - G \cdot X| \le l\left|\frac{Z_2}{Z_2 + Z_1 + S_0}\right|\varepsilon \le l\varepsilon.$$

Then from Lemmas 3.1 and 3.2, the washout equilibrium $E_1(1, 0)$ of model (3.4) is stochastically asymptotically stable.

Therefore when t is really large, we have $S_1 = 1$, $X_1 = 0$ which flows into the second reactor of the cascade. It is similar with the first reactor. Easily we can get the washout equilibrium $E_2(1, 0)$ is stochastically asymptotically stable. Similar arguments can prove $E_i(1, 0)$ is stochastically asymptotically stable. So when $\tau < \frac{1}{1-k_d}$, the washout equilibrium $E = (S_1, X_1, S_2, X_2, \ldots, S_n, X_n) = (1, 0, 1, 0, \ldots, 1, 0)$ of model (1.2) is stochastically asymptotically stable if σ satisfies $\sigma^2 \le \frac{2}{\tau} - 2 + 2k_d$. This completes the proof. □

This Theorem tells us that the washout equilibrium E is stable if the noise is bounded.

4 Asymptotic behavior of the noise perturbed non-washout equilibrium

Generally, E^* is not an equilibrium solution of the stochastic model (1.2) any more if $\sigma \neq 0$. As in this paper σ is small and we treat (1.2) as the perturbation of model (1.1) which has an non-washout equilibrium E^*, it is reasonable to consider the microorganism to be persist if solution of model (1.2) is going around E^* at the most of time. In this sense, investigating the asymptotic behavior still makes sense. And we have conclusion as follows.

Theorem 4.1 *Assume* $\sigma^2 \le \frac{1}{\tau} + k_d - \left(\frac{1}{\tau} + \frac{k_d}{2}\right)\frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{x_i^*}{x_i^* + s_i^* + 4\frac{x_i^*}{\alpha}}\right)$. *Then we have*

$$\lim_{t\to\infty}\sup\frac{1}{t}E\int_0^t\sum_{i=1}^n\left[(S_i(u)-S_i^*)^2+r_i^2(X_i(u)-X_i^*)^2\right]du\le k_\sigma,$$

where $(S_1(t), X_1(t), \ldots, S_n(t), X_n(t))$ *is a solution of model* (1.2) *associated with the initial value* $(S_1(0), X_1(0)) \in R_+^2$, *and*

$$r_i^2=\frac{-\frac{2}{\tau}+\max\left\{\left(\frac{1}{\tau}+\frac{k_d}{2}\right),\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\right\}}{2\sigma^2-\frac{2}{\tau}-2k_d+\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)}\alpha^2,$$

$$k_\sigma=\frac{c}{-2\sigma^2+\frac{2}{\tau}+2k_d-\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)},$$

$$c=\frac{2(\alpha+1)^2}{\tau}n+\sum_{i=1}^n2\sigma^2X_i^{*2}+\sum_{i=1}^n\left(\frac{2\alpha}{\tau}+\alpha k_d\right)\left(X_i^*+S_i^*\right)\sigma^2X_i^{*2}.$$

Proof Construct a Lyapunov function

$$V=\sum_{i=1}^n\left[\alpha\left(S_i-S_i^*\right)+\left(X_i-X_i^*\right)\right]^2$$
$$+\sum_{i=1}^n\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i^*+S_i^*\right)\left(X_i-X_i^*-X_i^*\ln\frac{X_i}{X_i^*}\right).$$

Differentiating it along the trajectory of (1.2) yields

$$dV=\sum_{i=1}^n2\left[\alpha\left(S_i-S_i^*\right)+\left(X_i-X_i^*\right)\right](\alpha dS_i+dX_i)+\sum_{i=1}^n\alpha(dS_i)^2+\sum_{i=1}^n(dX_i)^2$$
$$+\sum_{i=1}^n\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)(X_i^*+S_i^*)\left[\left(1-\frac{X_i^*}{X_i}\right)dX_i+\frac{X_i^*}{2X_i^2}(dX_i)^2\right]$$
$$=LVdt-\left[2\sigma\sum_{i=1}^nX_i\left(\alpha\left(S_i-S_i^*\right)+\left(X_i-X_i^*\right)\right)\right.$$
$$\left.+\sigma\sum_{i=1}^n\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i^*+S_i^*\right)\left(X_i-X_i^*\right)\right]dB(t),$$

where

$$
\begin{aligned}
LV &= \sum_{i=1}^{n} 2\left[\alpha\left(S_i - S_i^*\right) + \left(X_i - X_i^*\right)\right] \\
&\quad \times \left(\frac{\alpha}{\tau}\left(S_{i-1} - S_i\right) - \frac{1}{\tau}\left(X_{i-1} - X_i\right) - k_d X_i\right) \\
&\quad + \sum_{i=1}^{n} \sigma^2 X_i^2 + \sum_{i=1}^{n}\left(\frac{4\alpha}{\tau} + 2\alpha k_d\right)\left(X_i^* + S_i^*\right) \\
&\quad \times \left[\left(X_i - X_i^*\right)\left(\frac{1}{\tau X_i}\left(X_{i-1} - X_i\right) + \frac{S_i}{X_i + S_i} - k_d\right) + \frac{\sigma^2 X_i^*}{2}\right].
\end{aligned}
$$

Note that $-\frac{X_i^* X_i^*}{X_i^* + S_i^*} = \frac{1}{\tau} X_{i-1}^* - \frac{1}{\tau} X_i^* - k_d X_i^*$. Then we have

$$
\frac{\alpha}{\tau} S_{i-1}^* - \frac{\alpha}{\tau} S_i^* + \frac{1}{\tau} X_{i-1}^* - \frac{1}{\tau} X_i^* - k_d X_i^* = 0
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n}\left[2\alpha(S_i - S_i^*) + 2\left(X_i - X_i^*\right)\right]\left(\frac{\alpha}{\tau} S_{i-1} - \frac{\alpha}{\tau} S_i + \frac{1}{\tau} X_{i-1} - \frac{1}{\tau} X_i - k_d X_i\right) \\
&= \sum_{i=1}^{n}\left[2\alpha\left(S_i - S_i^*\right) + 2\left(X_i - X_i^*\right)\right]\left[\frac{\alpha}{\tau} S_{i-1} - \frac{\alpha}{\tau} S_i + \frac{1}{\tau} X_{i-1} - \frac{1}{\tau} X_i - k_d X_i\right. \\
&\quad \left. - \left(\frac{\alpha}{\tau} S_{i-1}^* - \frac{\alpha}{\tau} S_i^* + \frac{1}{\tau} X_{i-1}^* - \frac{1}{\tau} X_i^* - k_d X_i^*\right)\right] \\
&= \sum_{i=1}^{n}\left[-\frac{2\alpha^2}{\tau}(S_i - S_i^*)^2 + \left(-\frac{2}{\tau} - 2k_d\right)\left(X_i - X_i^*\right)^2 + \left(-\frac{4\alpha}{\tau} - 2\alpha k_d\right)\right. \\
&\quad \times \left(X_i - X_i^*\right)\left(S_i - S_i^*\right) + \left(2\alpha\left(S_i - S_i^*\right) + 2\left(X_i - X_i^*\right)\right)\frac{\alpha}{\tau}\left(S_{i-1} - S_{i-1}^*\right) \\
&\quad \left. + (2\alpha(S_i - S_i^*) + 2(X_i - X_i^*))\frac{1}{\tau}\left(X_{i-1} - X_{i-1}^*\right)\right] \\
&\le \sum_{i=1}^{n}\left[-\frac{2\alpha^2}{\tau}(S_i - S_i^*)^2 + \left(-\frac{2}{\tau} - 2k_d\right)(X_i - X_i^*)^2\right. \\
&\quad \left. + \left(-\frac{4\alpha}{\tau} - 2\alpha k_d\right)\left(X_i - X_i^*\right)(S_i - S_i^*) + \frac{2(\alpha + 1)^2}{\tau} n\right],
\end{aligned}
$$

which imply the following holds.

$$\sum_{i=1}^{n}\sigma^2X_i^2=\sum_{i=1}^{n}\sigma^2(X_i-X_i^*+X_i^*)^2 \\ \le\sum_{i=1}^{n}\left[2\sigma^2(X_i-X_i^*)^2+2\sigma^2X_i^{*2}\right].$$

Given the facts that $X_i^* \in [0, 1]$ and $S_i^* \in [0, 1]$, $S_i(t)$ and $X_i(t)$ are real numbers satisfing $0 < S_i(t) \le 1$, $0 < X_i(t) \le 1$, we have

$$\sum_{i=1}^{n}\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i^*+S_i^*\right)\left(X_i-X_i^*\right)\left(\frac{1}{\tau X_i}(X_{i-1}-Xi)+\frac{S_i}{X_i+S_i}-k_d\right) \\ =\sum_{i=1}^{n}\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)(X_i^*+S_i^*)\left(X_i-X_i^*\right)\left(\frac{S_i}{X_i+S_i}-\frac{S_i^*}{X_i^*+S_i^*}\right) \\ =\sum_{i=1}^{n}\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\left(\frac{X_i^*S_i-S_i^*X_i}{X_i+S_i}\right).$$

We will claim that for all solutions $(S_i(t), X_i(t))$ with initial value $(S_i(0), X_i(0)) \in R_+^2$,

$$LV\le\sum_{i=1}^{n}\left[-\frac{2}{\tau}+\max\left\{\left(\frac{1}{\tau}+\frac{k_d}{2}\right),\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\right\}\right]\alpha^2(S_i-S_i^*)^2 \\ +\sum_{i=1}^{n}\left[2\sigma^2-\frac{2}{\tau}-2k_d+\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\right](X_i-X_i^*)^2+c. \tag{4.1}$$

Notice that

$$\sum_{i=1}^{n}\left(-\frac{4\alpha}{\tau}-2\alpha k_d\right)\left(X_i-X_i^*\right)(S_i-S_i^*) \\ +\sum_{i=1}^{n}\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\left(\frac{X_i^*S_i-S_i^*X_i}{X_i+S_i}\right) \\ =\sum_{i=1}^{n}\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\frac{S_i(X_i^*+S_i^*-S_i-X_i)}{X_i+S_i}. \tag{4.2}$$

First, we prove for each i,

$$\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\frac{S_i(X_i^*+S_i^*-S_i-X_i)}{X_i+S_i}$$
$$\le \max\left\{\left(\frac{1}{\tau}+\frac{k_d}{2}\right),\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\right\}\alpha^2(S_i-S_i^*)^2$$
$$+\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)(X_i-X_i^*)^2.$$

Suppose first $(X_i-X_i^*)(S_i-S_i^*)\ge 0$, which contains three cases.

(a) $X_i-X_i^*>0$, $S_i-S_i^*>0$. In this case, Eq. (4.2) is negative;
(b) $X_i-X_i^*<0$, $S_i-S_i^*<0$. This case also implies Eq. (4.2) is negative;
(c) either $X_i-X_i^*=0$ or $S_i-S_i^*=0$. In this case the Eq. (4.2) is non-positive.

In other words, when $(X_i-X_i^*)(S_i-S_i^*)\ge 0$ Eq. (4.1) is true. Next, we prove it is also true when $(X_i-X_i^*)(S_i-S_i^*)<0$. we have two cases.

Case 1): $X_i-X_i^*>0$, $S_i-S_i^*<0$.
If $X_i^*+S_i^*\le S_i+X_i$, (4.2) is nonpositive. Then we can get that (4.1) is true. Otherwise, we have $X_i^*+S_i^*>S_i+X_i$. Then we have

$$\left(-\frac{4\alpha}{\tau}-2\alpha k_d\right)\left(X_i-X_i^*\right)\left(S_i-S_i^*\right)+\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)$$
$$\times\left(X_i-X_i^*\right)\left(\frac{X_i^*S_i-S_i^*X_i}{X_i+S_i}\right)<\left(-\frac{4\alpha}{\tau}-2\alpha k_d\right)\left(X_i-X_i^*\right)\left(S_i-S_i^*\right)$$
$$+\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\left(\frac{X_i^*(S_i-S_i^*)}{X_i^*+S_i^*}\right)$$
$$=\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(\frac{X_1^*}{X_i^*+S_i^*}-1\right)\left(X_i-X_i^*\right)\left(S_i-S_i^*\right)$$
$$=\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*}\right)\left|X_i-X_i^*\right|\left|S_i-S_i^*\right|$$
$$\le\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*}\right)\left[(X_i-X_i^*)^2+\alpha^2(S_i-S_i^*)^2\right]$$
$$=\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*}\right)\alpha^2(S_i-S_i^*)^2$$
$$+\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*}\right)(X_i-X_i^*)^2.$$

Case 2): $X_i-X_i^*<0$, $S_i-S_i^*>0$.

If $X_i^*+S_i^*\geq S_i+X_i$, the (4.2) is nonpositive. Then (4.1) holds. Otherwise, $X_i^*+S_i^*<S_i+X_i$. If $S_i>S_i^*+4\frac{X_i^*}{\alpha}$, we can get $X_i^*<\frac{\alpha(S_i-S_i^*)}{4}$.

$$\begin{aligned}
&\left(-\frac{4\alpha}{\tau}-2\alpha k_d\right)\left(X_i-X_i^*\right)\left(S_i-S_i^*\right)\\
&=\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i^*-X_i\right)\left(S_i-S_i^*\right)\\
&\leq\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)(S_i-S_i^*)X_i^*\\
&\leq\left(\frac{\alpha^2}{\tau}+\frac{\alpha^2 k_d}{2}\right)(S_i-S_i^*)^2\\
&=\left(\frac{1}{\tau}+\frac{k_d}{2}\right)\alpha^2(S_i-S_i^*)^2,
\end{aligned}$$

while

$$\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\left(\frac{X_i^*S_i-S_i^*X_i}{X_i+S_i}\right)\leq 0.$$

If $S_i\leq S_i^*+4\frac{X_i^*}{\alpha}$,

$$\begin{aligned}
&\left(-\frac{4\alpha}{\tau}-2\alpha k_d\right)\left(X_i-X_i^*\right)\left(S_i-S_i^*\right)+\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\\
&\times\left(\frac{X_i^*S_i-S_i^*X_i}{X_i+S_i}\right)\leq-\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(X_i-X_i^*\right)\left(S_i-S_i^*\right)\\
&\quad+\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\left(X_i-X_i^*\right)\left(S_i-S_i^*\right)\\
&=\left(\frac{4\alpha}{\tau}+2\alpha k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\left|X_i-X_i^*\right|\left|S_i-S_i^*\right|\\
&\leq\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\left(\left(X_i-X_i^*\right)^2+\alpha^2\left(S_i-S_i^*\right)^2\right)\\
&=\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\alpha^2\left(S_i-S_i^*\right)^2\\
&\quad+\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\left(X_i-X_i^*\right)^2
\end{aligned}$$

Therefore

$$LV \leq \sum_{i=1}^{n}\left[-\frac{2}{\tau}+\max\left\{\left(\frac{1}{\tau}+\frac{k_d}{2}\right),\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\right\}\right]\alpha^2(S_i-S_i^*)^2$$
$$+\sum_{i=1}^{n}\left[2\sigma^2-\frac{2}{\tau}-2k_d+\left(\frac{2}{\tau}+k_d\right)\left(1-\frac{X_i^*}{X_i^*+S_i^*+4\frac{X_i^*}{\alpha}}\right)\right](X_i-X_i^*)^2+c.$$

where c is defined as before.

It then follows from

$$E\int_0^t dV = E\int_0^t LV dt,$$

that

$$\lim_{t\to\infty}\sup\frac{1}{t}E\int_0^t\sum_{i=1}^{n}\left[\left(S_i(u)-S_i^*\right)^2+r_i^2\left(X_i(u)-X_i^*\right)^2\right]du \leq k_\sigma,$$

where r_i^2 and k_σ are defined in the theorem statement. This completes the proof. □

5 Numerical simulation and discussion

In this section, we carry out numerical simulations to demonstrate the stochastic stability of the equilibrium solutions. We choose a cascade of four reactors. Our simulations agree well with our theoretical analysis in previous Sects. 3 and 4. Performance of the bioreactor can be affected by noise in certain degree.

5.1 Stochastic stability of the washout and non-washout equilibrium

Choose $(0.5, 0.5)$ as the initial value and set parameters as $\alpha = 1, S_0 = 1, k_d = 0.6$, $\tau = 2$ and $\sigma = 0.2$. Then we can verify that $\tau < \frac{1}{1-k_d}$ and $\sigma^2 \leq \frac{2}{\tau}-2+2k_d = 0.2$. By using results from Sect. 3 we know that the equilibrium $(1, 0, 1, 0, 1, 0, 1, 0)$ should be globally asymptotically stable. Our simulations are shown in Fig. 1, which agree well with the theoretical result. As seen, the effect of the noise on the stability of the washout equilibrium is getting more and more obvious as σ^2 increases.

Next we demonstrate the non-washout equilibrium is stochastically stable too. We choose $\alpha = 1, k_d = 0.2, \tau = 2$. After some basic calculation, we know model (1.1) has a unique positive equilibrium $(0.625, 0.268, 0.282, 0.436, 0.109, 0.436, 0.039, 0.361)$, and

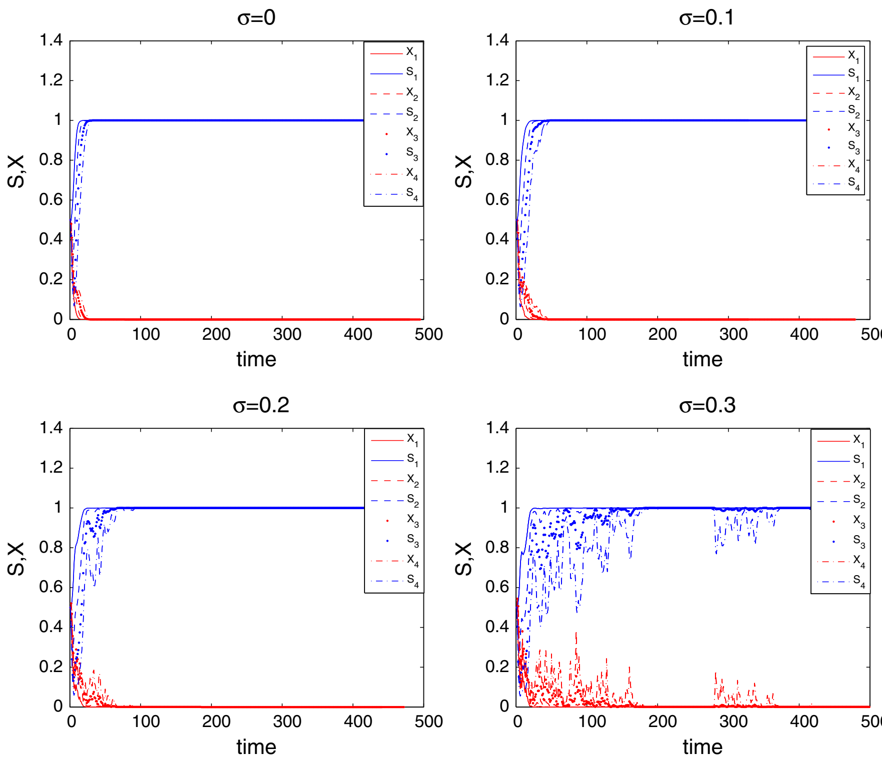

Fig. 1 Stability of E_1. Comparison of the dynamics in deterministic model and stochastic model with $\sigma = 0.1, 0.2, 0.4$, respectively

$$\frac{1}{\tau} + k_d - \left(\frac{1}{\tau} + \frac{k_d}{2}\right) \frac{1}{4} \sum_{i=1}^{4} \left(1 - \frac{x_i^*}{x_i^* + s_i^* + 4\frac{x_i^*}{\alpha}}\right) = 0.205.$$

Then from Sect. 4 the non-washout equilibrium is stable when $\sigma^2 \leq 0.205$. $(0.66, 0.18)$ is chosen as the initial value and simulation are listed in Fig. 2. In Fig. 3, we show the case $\sigma^2 > 0.205$, in which as seen the non-washout equilibrium is unstable.

5.2 Performance of the bioreactor

In order to analyse the performance of reactor cascades, we introduce the following dimensionless quantities: the **specific utilisation**

$$\mathcal{U} = \frac{S_0 - S_n^*}{X_n^*} \frac{1}{\tau}, \tag{5.1}$$

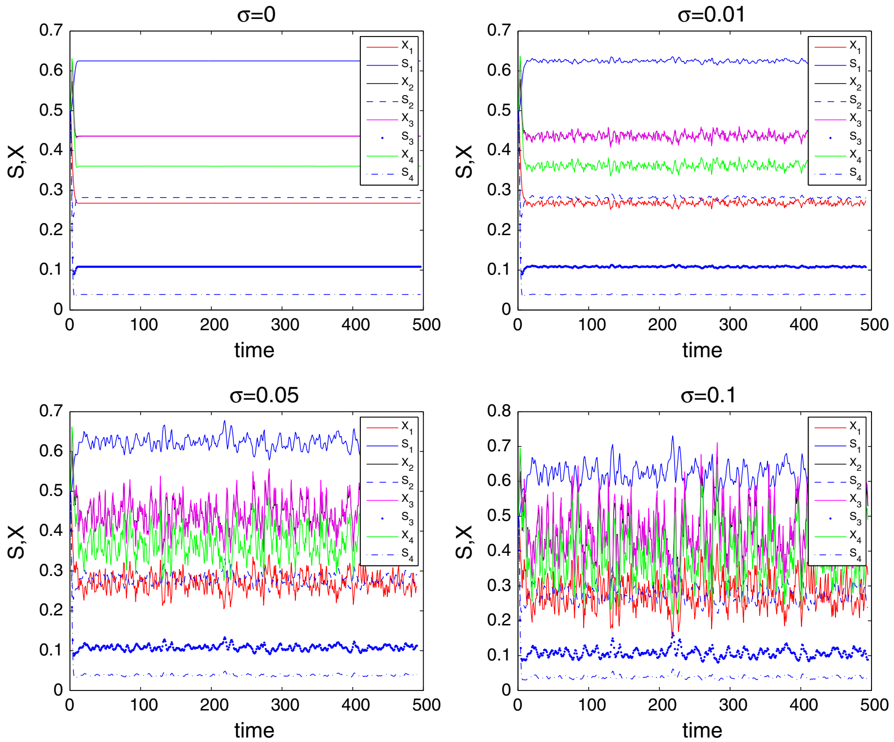

Fig. 2 Stability of E_2. Comparison of the dynamics in deterministic model and stochastic model with $\sigma = 0.01, 0.05, 0.1$, respectively

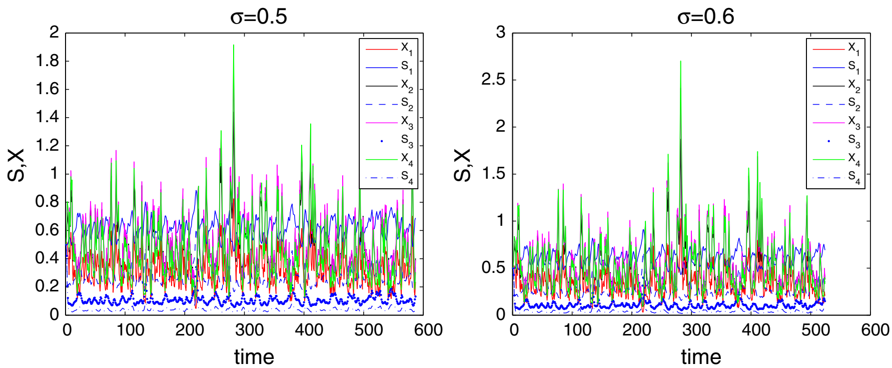

Fig. 3 Instability of E_2. The long time behavior of the stochastic model with $\sigma = 0.5, 0.6$ respectively

the **yield**

$$\mathcal{Y} = \frac{X_n^*}{S_0 - S_n^*}, \tag{5.2}$$

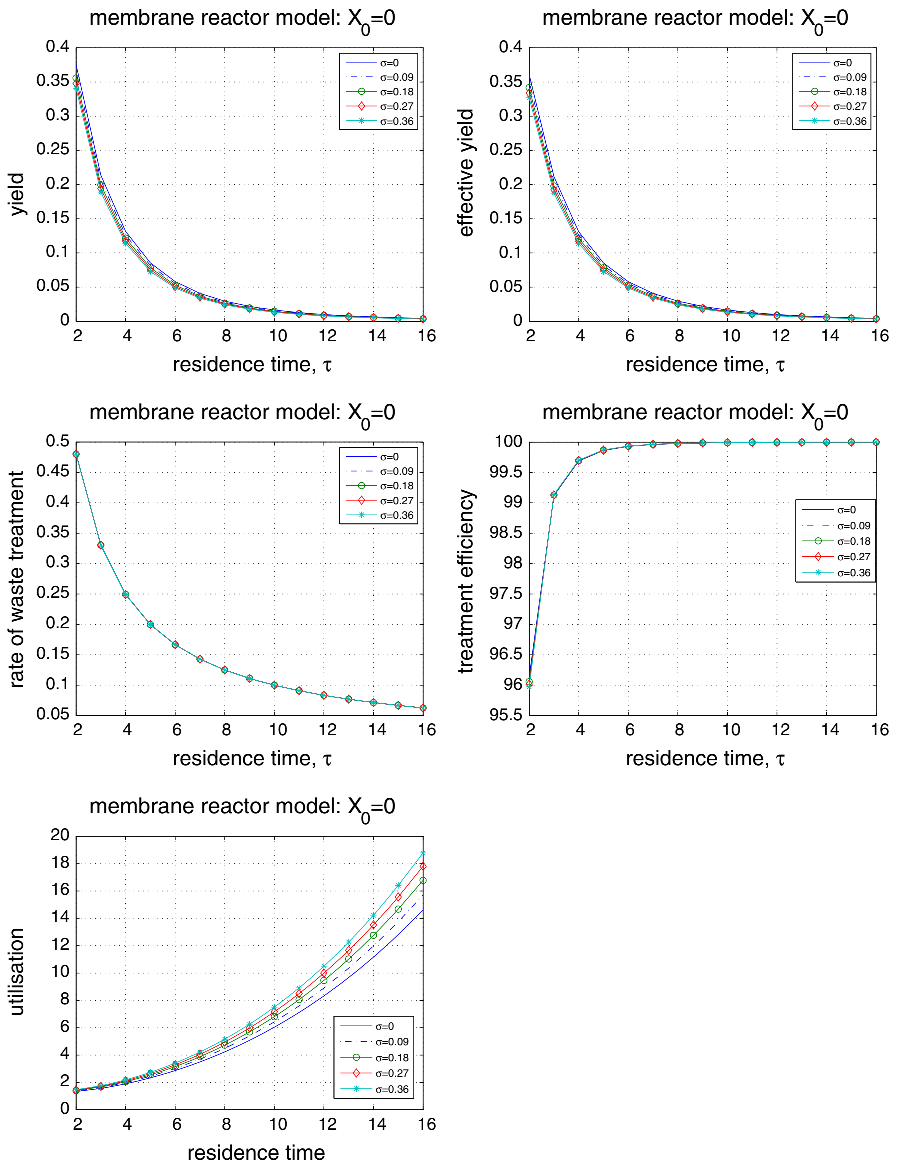

Fig. 4 Performance of the bioreactor

the **treatment/process efficiency**

$$\mathcal{E} = 100 \times \frac{S_0 - S_n^*}{S_0}, \tag{5.3}$$

the **rate of waste treatment**

$$\mathcal{W} = \frac{S_0 - S_n^*}{\tau} \tag{5.4}$$

and the **effective yield**

$$\mathcal{Y}_e = \frac{X_n^*}{S_0}. \tag{5.5}$$

For more details about these quantities please see [10] and the references therein.

To show the effects of the performance of reactor cascades by the noise, we also selected a cascade of four reactors. Figure 4 shows the performance of the fourth reactor. It suggests that the yield, $\mathcal{Y}$, effective yield ($\mathcal{Y}_e$) and rate of waste treatment ($\mathcal{W}$) all decrease as σ increases from zero; however, the uilization ($\mathcal{U}$) increases with the noise intensity σ and the treatment ($\mathcal{E}$) decreases when σ increases from zero at first and then increases as it gets bigger, which implies there is certain value of $\sigma > 0$ which corresponds a minimum performance of the reactor, see the subfigures 2–4 of Fig. 4. Also Fig. 4 indicates that both the utilisation and the treatment efficiency are increasing functions of the residence time, τ; but yield, rate of waste treatment and effective yield are decreasing functions of τ. In other words, larger residence time results in a higher utilisation and the treatment efficiency, but lower yield, rate of waste treatment and effective yield.

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